The Spectral Theorem from a nonstandard perspective

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Descriptive Dynamics and Combinatorics Seminar

Storyline

Relevant objects

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- H, separable Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$).
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We assume, unless specified:

- H is infinite-dimensional.
- A is densely-defined.

The Spectral Theorem for self-adjoint operators

This is our object of study:

Theorem (Spectral Theorem)

If A is self-adjoint, then A is unitarily equivalent to a multiplication operator. In other words, there exists a measure space $(\Omega, \mathcal{A}, \mu)$, a measurable function $\lambda : \Omega \to \mathbb{R}$ and a unitary map $U : H \to L_2(\Omega, \mu)$ such that for any $x \in \mathrm{dom}(A)$, $U(Ax) = \lambda \cdot U(x)$.

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But why?

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$$\|U(x)\|^2 = \int_\Omega |U(x)|^2 d\mu = \sum_{f \in \Omega} \left| \frac{\langle x, f \rangle}{\sqrt{\mu(\{f\})}} \right|^2 \mu(\{f\}) = \|x\|^2$$

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- Multiplication operator given by eigenvalue function $\lambda : \Omega \to \mathbb{R}$

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1. $H_n < H$ and $\dim(H_n) < \infty$ for each $n \in \mathbb{N}$;
2. $A_n : H_n \to H_n$ is a symmetric linear operator for each n;
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4. for each $x \in \mathrm{dom}(A)$, there exists a sequence $(x_n \in H_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $A_n x_n \to Ax$.

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One can prove: there always exists a sampling sequence (not hard, using that $G(A) \subset H \times H$ is separable)

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- One way (not the only one!): Consider one single "infinitely good" "finite dimensional" approximation, and work from its induced measure

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- $^*V$ may be much, much larger than V, for any infinite V.

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- for every finite $x \in {}^*\mathbb{R}$, there exists a unique $s \in \mathbb{R}$ such that $x \simeq s$.
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- This defines the standard part function $\mathrm{st} : \mathrm{Fin}({}^*\mathbb{R} \to \mathbb{R})$
- If (X, d) is metric space, $x \in {}^*X$ is nearstandard if there is $s \in X$ such that ${}^*d(x, s) \simeq 0$; we define $s = \mathrm{st}(x)$.

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- $\lim_{n\to\infty} x_n = x$ iff for any infinite $N \in {}^*\mathbb{N}$, ${}^*x_N \simeq x$
- f is continuous at x_0 iff ${}^*f(x) \simeq f(x_0)$ whenever $x \simeq x_0$
- X is compact iff every $x \in {}^*X$ is nearstandard

Internal sets

- $(V \cup \mathcal{P}(V), \in)$ is itself a (relational) structure, on which we can apply $*$
- We have a natural inclusion ${}^*\mathcal{P}(V) \subset \mathcal{P}({}^*V)$ with $S \leftrightarrow \{x \in {}^*V \mid x \ ({}^* \in) S\}$
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- Some internal sets: ${}^*S$, any finite $F \subset {}^*V$, $\{n \in {}^*N \mid n \leq N\}$ given some infinite $N \in {}^*\mathbb{N}$, any set defined with only internal objects and the language of V.
- Some external sets: $\mathbb{N}$ (or any infinite set with only standard elements), { infinite hypernatural }, $\{x \in {}^*\mathbb{R} \mid \text{st}(x) = 3\}$

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- Some external sets: $\mathbb{N}$ (or any infinite set with only standard elements), { infinite hypernatural }, $\{x \in {}^*\mathbb{R} \mid \operatorname{st}(x) = 3\}$
- Transfer Principle: transfers only for internal objects, ex:

$$\forall S \in \mathcal{P}(\mathbb{N})((1 \in S \wedge \forall n \in \mathbb{N}(n \in S \rightarrow n+1 \in S)) \rightarrow S = \mathbb{N})$$

Hyperfinite

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- ex: $\{n \in {}^*\mathbb{N} \mid n \leq N\}$ for any $N \in {}^*\mathbb{N}$
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- Enlargement Theorem: There exists an extension * and hyperfinite V_F such that $V \subset V_F \subset {}^*V$ regardless of $|V|$

Nonstandard sampling

Given sampling sequence $(H_n, A_n, \Omega_n)_{n \in \mathbb{N}}$ and infinite $N \in {}^*\mathbb{N}$, let $(\tilde{H}, \tilde{A}, \tilde{\Omega})$ be given by $\tilde{H} = {}^*H_N$, $\tilde{A} = {}^*A_N$, $\tilde{\Omega} = {}^*\Omega_N$.

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- $\tilde{H}$ is a ${}^*\mathbb{K}$ subspace of ${}^*H$, and ${}^*\dim(\tilde{H}) \in {}^*\mathbb{N}$
- $\tilde{A}$ is an internal symmetric operator on $\tilde{H}$
- $\tilde{\Omega}$ is a (hyperfinite) orthnormal eigenbasis of $\tilde{A}$
- For any $x \in \mathrm{dom}(A)$, there exists $\tilde{x} \in \tilde{H}$ such that $x = \mathrm{st}(\tilde{x})$ and $Ax = \mathrm{st}(\tilde{A}\tilde{x})$.

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- $\tilde{\mu} = {}^*\mu_N$ is an internal measure on the internal algebra $\tilde{\mathcal{A}} = {}^*\mathcal{P}(\tilde{\Omega})$
- $\tilde{U} : \tilde{H} \to {}^*L_2(\tilde{\Omega}, \tilde{\mu})$ is an internal unitary equivalence between $\tilde{A}$ and $\tilde{\lambda} = {}^*\lambda_N$

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We use the powerful Loeb measure Theorem:

Theorem (Loeb measure Theorem)

There exists a (real, external) probability space $(\tilde{\Omega}, \mathcal{A}_L, \mu_L)$ *such that*

- $\tilde{\mathcal{A}} \subset \mathcal{A}_L$
- *for any* $B \in \mathcal{A}$, $\mu_L(B) = \mathrm{st}(\tilde{\mu}(B))$.

Problem 1

- We consider $U_L : H \to L_2(\tilde{\Omega}, \mu_L)$ with $U_L(x) = \text{st} \circ \tilde{U}(x)$.
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- We want U_L as our isometry. For that, we need

$$\mathrm{st}\left({}^{*}\int_{\tilde{\Omega}} |\tilde{U}(x)|^2 d\tilde{\mu} \right) = \int_{\tilde{\Omega}} \mathrm{st}(|\tilde{U}(x)|^2) d\mu_L$$

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It does not always hold (some dirac-like function?)

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Theorem (S-integrability)

For any internal function $f : \tilde{\Omega} \to {}^*\mathbb{R}_{\geq 0}$*, the following are equivalent:*

- *for any internal* $E \subset \tilde{\Omega}$ *with* $\tilde{\mu}(E) \simeq 0$, ${}^*\int_E f d\tilde{\mu} \simeq 0$
- f *is* μ_L *almost-always nearstandard valued, and* $\operatorname{st}\left(\int_{\tilde{\Omega}} f d\tilde{\mu}\right) = \int_{\tilde{\Omega}} \operatorname{st} \circ f d\mu_L$

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- We need that $\tilde{\mu}(E) \simeq 0 \implies$ for all standard $x \in H$, ${}^* \operatorname{proj}_{\operatorname{span}(E)} x \simeq 0$

Problem 2

- We want $\tilde{\lambda} : \tilde{\Omega} \to {}^*\mathbb{R}$, the eigenvalue function of $\tilde{A}$, to be μ_L almost always finite,
- Equivalently, for every infinite K, we want $\tilde{\mu}(B_K) \simeq 0$ with $B_K = \{f \in \tilde{\Omega} \mid |\tilde{\lambda}(f)|^2 \geq K\}$

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- Thus, for any standard $x \in H$, ${}^* \operatorname{proj}_{\operatorname{span}(B_K)} x \simeq 0$
- It would be nice if $\tilde{\mu}(B_K) \simeq 0$ followed

Standard Bias measure

Definition

The internal probability measure $\tilde{\mu}$ on ${}^*\mathcal{P}(\tilde{\Omega})$ is standard-biased if for any internal $E \in \tilde{\Omega}$, $\tilde{\mu}(E) \simeq 0$ if and only if ${}^*\operatorname{proj}_{\operatorname{span}(E)} x \simeq 0$ whenever $x \in H$ is standard.

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- For such a standard biased measure, we thus consider $\lambda_L = \operatorname{st} \circ \tilde{\lambda}$

Compatible standard-biased scale

There always exists $((\tilde{e}_j)_{j=1}^K, (\tilde{c}_j)_{j=1}^K)$, where

- $K \in {}^*\mathbb{N}$ is infinite, $\tilde{e}_j \in {}^*H$ and $\tilde{c}_j \in {}^*\mathbb{R}_{>0}$
- For any standard j, $\mathrm{st}(\tilde{e}_j) = e_j \in H \setminus 0$, $\mathrm{st}(\tilde{c}_j) = c_j \in \mathbb{R}_{>0}$

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- $(e_j)_{j\in\mathbb{N}}$: dense span in H
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For compatibility with sampling $(\tilde{H}, \tilde{A}, \tilde{\Omega})$:

- For any $j \leq N$, $\tilde{e}_j \in \tilde{H}$
- For any standard j, $\tilde{A}\tilde{e}_j$ is nearstandard
- For any $f \in \tilde{\Omega}$, there exist $j \leq N$ with $\langle \tilde{e}_j, f \rangle \neq 0$

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Induces standard-biased probability internal measure $\tilde{\mu}$ on $\tilde{\Omega}$ with $\tilde{\mu}(V) = \sum_{j=1}^K \tilde{c}_j \|{}^* \mathrm{proj}_{{}^*\mathrm{span}(V)}\, \tilde{e}_j\|^2$

The Spectral Embedding Theorem

We can then establish the following:

Theorem (The Spectral Embedding Theorem)

If $\tilde{\mu}$ is standard-biased, then $U_L : H \to L_2(\tilde{\Omega}, \mu_L)$ is an isometry. Furthermore, for any $x \in \mathrm{dom}(A)$, $U_L(Ax) = \lambda_L \cdot U_L(x)$.

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- Directly equivalent with other forms of the Spectral Theorem: if A is self-adjoint, $U_L(H)$ reduces for the multiplication operator induced by λ_L, and spectral measure of A pulls back
- $(\tilde{\Omega}, \mathcal{A}_L, \mu_L)$ still heavily dependent on both * and infinite N, and $L_2(\Omega_L, \mu_L)$ non-separable. This needs a sequel...

Internal pseudometric

- Using the standard-biased scale, we can construct internal pseudometric $\tilde{d}$ on $\tilde{\Omega}$ such that $\tilde{d}(f_1, f_2) \simeq 0$ iff $(\tilde{U}(\tilde{e}_j))(f_1) \simeq (\tilde{U}(\tilde{e}_j))(f_2)$ for all standard j

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- $$\tilde{d}(f_1, f_2) = \sum_{j=1}^{K} \tilde{c}_j^{\frac{3}{2}} \|\tilde{e}_j\|^2 |(\tilde{U}(\tilde{e}_j))(f_1) - (\tilde{U}(\tilde{e}_j))(f_2)|$$

Hull space

- $\hat{\nu}$ is measurable (w.r.t. μ_L), inducing the pushforward probability space $(\hat{\Omega}, \text{Borel}(\hat{\Omega}), \hat{\mu})$
- all elements of $U_L(H)$ can be pushed down $\hat{\nu}$, inducing isometry $\hat{U} : H \to L_2(\hat{\Omega}, \hat{\mu})$
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Thus, we have the following:

Theorem (Spectral Embedding Theorem, metric version)

For any densely-defined symmetric operator A on separable $\mathbb{K}$-Hilbert space H, there exists a compact metric space Ω, a probability measure μ on $\text{Borel}(\Omega)$, an isometric embedding $U : H \to L_2(\Omega, \mu)$ and a self-adjoint multiplication operator T on $L_2(\Omega, \mu)$ such that $U \circ A \subset T \circ U$.

Shift Operator

Context

- $H = l_2(\mathbb{Z})$, $\mathbb{K} = \mathbb{C}$ and $(g_l)_{l \in \mathbb{Z}}$ canonical Hilbert basis
- $A = \frac{1}{2}(R + L)$, where R and L are the right and left shifts.

Shift Operator

Parameters:

- $\tilde{H} = {}^*\text{span}(\{{}^*g_l\}_{l=-M}^{M})$ for some infinite M
- $\tilde{A} = \frac{1}{2}(\tilde{R} + \tilde{L})$, where $\tilde{R}$ and $\tilde{L}$ are the rotating right and left shifts on $\tilde{H}$

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- $(\tilde{e}_j)_{j=1}^{2M+1} = ({}^*g_0, {}^*g_1, {}^*g_{-1}, \ldots, {}^*g_M, {}^*g_{-M})$
- $\tilde{c}_j = \frac{1}{2^j(1-2^{-(2M+1)})}$ for $j \leq 2M+1$

Shift Operator

Results (up to measure-preserving homeomorphism):

- $\Omega = \mathbb{R}/\mathbb{Z}$, equipped with its usual topology
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- $U((a_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{N}} a_n e^{2\pi i n \cdot}$, associating a sequence to its Fourier series

Differential operator on $\mathbb{R}$

Context:

- $H = L_2(\mathbb{R})$, with $\mathbb{K} = \mathbb{C}$
- $A = -i\frac{d}{dx}$ on $\text{dom}(A) = C_c^\infty(\mathbb{R})$

Differential operator on $\mathbb{R}$

Parameters:

- $\tilde{H} = {}^*\mathrm{span}(\{1_{[\frac{k}{N},\frac{k+1}{N}[}\}_{k=-N^2}^{N^2-1})$ for some infinite $N = N_0$!
- $\tilde{A} = -i\frac{\tilde{L}-\tilde{R}}{2/N}$ given rotating "shift" operators
- $\tilde{\Omega} = \{f_k\}_{k=-N^2}^{N^2-1}$ with $f_k = \frac{1}{\sqrt{2N}}\sum_{l=-N^2}^{N^2-1} e^{2\pi i\frac{kl}{2N^2}} 1_{[\frac{l}{N},\frac{l+1}{N}[}$

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- Given $E(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} e^{-t^2}$ on $\mathbb{R}$, let $e_j(t) = E(t-q_j)$ for $(q_j)_{j\in\mathbb{N}}$ being a counting of $\mathbb{Q}$. Scale is constructed around this

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Results (up to measure space equivalence):

- $\Omega = \mathbb{R}$, equipped with its usual borelian algebra (not the same topology, though)
- $\mu' = g_0 d\mu$ with $g_0(\omega) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-\frac{\pi^2 \omega^2}{2}}$

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Of note:

- only elementary analysis used for calculations
- could theoretically be used to define the Fourier transform itself
- direct proofs of Plancherel and differentiation theorems

Conclusion

Thank you for your time!

Questions?